

Some Six-Dimensional Rigid Forms

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One can always decompose Dirichlet-Voronoi polytopes of lattices non-trivially into a Minkowski sum of Dirichlet-Voronoi polytopes of rigid lattices. In this report we show how one can enumerate all rigid positive semidefinite quadratic forms (and thereby rigid lattices) of a given dimension d . By this method we found all rigid positive semidefinite quadratic forms for $d = 5$ confirming the list of 7 rigid lattices by Baranovskii and Grishukhin. Furthermore, we found out that for $d \leq 5$ the adjacency graph of primitive L -type domains is an infinite tree on which $\mathrm{GL}_d(\mathbb{Z})$ acts. On the other hand, we demonstrate that in $d = 6$ we face a combinatorial explosion.

1. INTRODUCTION

Let L be a lattice in Euclidean space $(\mathbb{R}^d, \langle \cdot, \cdot \rangle)$. With L one associates the *Dirichlet-Voronoi polytope*

$$\mathrm{DV}(L) = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{x} \rangle \leq \langle \mathbf{v} - \mathbf{x}, \mathbf{v} - \mathbf{x} \rangle \text{ for all } \mathbf{v} \in L\}.$$

One can always decompose Dirichlet-Voronoi polytopes of lattices non-trivially into a Minkowski sum of Dirichlet-Voronoi polytopes of so-called rigid lattices. In this paper we show how one can find all rigid lattices in a given dimension.

It will be more convenient to use the language of quadratic forms. With L we associate a positive definite quadratic form: By choosing a lattice basis $\mathbf{b}_1, \dots, \mathbf{b}_d$ we get a positive definite matrix $Q = (\langle \mathbf{b}_i, \mathbf{b}_j \rangle)_{i,j}$ giving the positive definite quadratic form $\mathbf{x} \mapsto \mathbf{x}^t Q \mathbf{x}$.

By applying basic facts of Voronoi's theory of L -type domains we get an algorithm for finding all rigid forms of a given dimension. All quadratic forms whose Dirichlet-Voronoi polytopes have the same combinatorial-metric structure belong to a pointed polyhedral cone, a so-called L -type domain. L -types domains of maximal dimension are called *primitive*, L -type domains of minimal dimension 1 are called *rigid*, and elements of rigid L -type domains are called *rigid forms*. From Voronoi's algorithm for finding all primitive L -type domains we get the facets of every primitive L -type domain. By converting the facet description we find all extreme rays. If an L -type domain is invariant under a non-trivial symmetry group we can speed up the task of converting considerably using the adjacency decomposition method.

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We applied this algorithm for dimensions ≤ 5 . Thereby, we confirmed the results by Baranovskii and Grishukhin (2): There is exactly one 1-dimensional rigid form (the Dirichlet-Voronoi polytope is a line segment), there are no 2- and 3-dimensional rigid forms, there is exactly one 4-dimensional rigid form (the Dirichlet-Voronoi polytope is the 24-cell), and there are seven 5-dimensional rigid forms. Using this classification we verified that for $d \leq 5$ the graph of primitive L -type domains is an infinite tree on which the group $\mathrm{GL}_d(\mathbb{Z})$ acts. It is computationally quite simple to perform these classifications because face lattices of these primitive L -type domains are very tame.

This is no longer the case in dimension 6. We explored two notable primitive L -type domains of 6-dimensional quadratic forms. The first cone has 130 facets and we do not know a primitive L -type domain in dimension 6 having more facets. The second cone has 100 facets and it contains a positive definite quadratic form associated to the lattice whose covering density is conjectured to be optimal in dimension 6. The automorphism groups of both cones are fairly big. By using the adjacency decomposition method we succeeded to compute the extreme rays of both cones: The first one has 7,145,429 extreme rays, and the second one has 2,257,616 extreme rays. But many of these extreme rays correspond to equivalent (under the group $\mathrm{GL}_d(\mathbb{Z})$) rigid forms. In total we found 25,263 non-equivalent rigid 6-dimensional positive definite quadratic forms.

This paper is organized as follows: In Section 2 we recall some definitions and facts of Voronoi's theory of L -type domains. In Section 3 we show in which sense rigid forms are building blocks of Dirichlet-Voronoi polytopes. In Section 4 we explain the adjacency decomposition method. In Section 5 and Section 6 we report on computational results for rigid forms up to dimension 6.

2. NOTATION: L -TYPE DOMAINS

In this section we define L -type domains for positive semidefinite quadratic forms. This enables us to define rigid forms.

By \mathcal{S}^d we denote the space of all quadratic forms in d variables, and by $\mathcal{S}_{\geq 0}^d$ we denote the set of all positive semidefinite quadratic forms which is a closed pointed cone. In the following we will identify \mathcal{S}^d with the space of all symmetric $(d \times d)$ -matrices. We say that two quadratic forms Q, Q' are *arithmetically equivalent* if there exists an integral unimodular matrix $A \in \mathrm{GL}_d(\mathbb{Z})$ so that $Q' = A^t Q A$.

Let $Q \in \mathcal{S}_{\geq 0}^d$ be a positive semidefinite quadratic form arithmetically equivalent to $\begin{pmatrix} Q' & 0 \\ 0 & 0 \end{pmatrix}$ where Q' is positive definite. We define the *Dirichlet-Voronoi polytope* of Q by

$$\mathrm{DV}(Q) = \{\mathbf{x}^t Q \in (\mathbb{R}^d)^* : \mathbf{x}^t Q \mathbf{x} \leq (\mathbf{x} - \mathbf{v})^t Q (\mathbf{x} - \mathbf{v}) \text{ for all } \mathbf{v} \in \mathbb{Z}^d\}.$$

This way of defining Dirichlet-Voronoi polytopes of positive semidefinite quadratic forms is due to Namikawa (7). The definition has the important

feature that we can define L -type domains of positive semidefinite quadratic forms by using the concept of strongly isomorphic polytopes which we recall now.

Let V be a finite dimensional real vector space. Let $P \subseteq V$ be a convex polytope, and let $f \in V^*$ be a linear functional. We define the *support functional* of P by $\eta(P, f) = \max\{f(x) : x \in P\}$ and by $P_f = \{x \in P : f(x) = \eta(P, f)\}$ we denote the face of P in direction f . We say that two convex polytopes $P, Q \in V$ are *strongly isomorphic* if for every $f, g \in V^*$ with $P_f \subseteq P_g$ we have $Q_f \subseteq Q_g$.

We say that two positive semidefinite quadratic forms Q, Q' belong to the same L -type if their Dirichlet-Voronoi polytopes are strongly isomorphic. The set of all positive semidefinite quadratic forms belonging to the same L -type is called an L -type domain. Two L -type domains Δ, Δ' are *arithmetically equivalent* if there exists $A \in \text{GL}_d(\mathbb{Z})$ so that $\Delta' = A^t \Delta A$.

In (11) Voronoi showed that L -type domains are open pointed polyhedral cones, that the L -type domains give a face-to-face partition of $\mathcal{S}_{\geq 0}^d$, and that there are only finitely many non-equivalent L -type domains. L -type domains which are of maximal dimension $\frac{d(d+1)}{2}$ are called *primitive*. L -type domains which are of minimal dimension 1 are called *rigid*. Positive semidefinite quadratic forms lying in a rigid L -type domain are called *rigid*, too.

3. RIGID FORMS AND DIRICHLET-VORONOI POLYTOPES

In this section we show in what sense Dirichlet-Voronoi polytopes of rigid positive semidefinite quadratic forms are building blocks of Dirichlet-Voronoi polytopes of general positive semidefinite quadratic forms.

Lemma 1. *Every Dirichlet-Voronoi polytope of a positive semidefinite quadratic form is Minkowski sum of Dirichlet-Voronoi polytopes of rigid forms. More precisely: Let $\overline{\Delta}$ be the topological closure of an L -type domain. For positive semidefinite quadratic forms $Q_1, \dots, Q_n \in \overline{\Delta}$ and non-negative numbers $\alpha_1, \dots, \alpha_n$ we have*

$$\text{DV}\left(\sum_{i=1}^n \alpha_i Q_i\right) = \sum_{i=1}^n \alpha_i \text{DV}(Q_i).$$

The authors do not know exactly the origin of this lemma. Loesch gave it in a dual formulation in (6). Later, Ryshkov gave in (9) a similar but less precise statement.

4. COMPUTATIONAL TECHNIQUES

Fukuda's program `cdd` (5) computes the list of extreme rays of a polyhedral cone given its list of facets. In our case the number of extreme rays can be very large so that we cannot use `cdd` naively. We apply another technique called

adjacency decomposition method to use the symmetry of the polyhedral cones we are considering.

Let $C \subseteq \mathbb{R}^d$ be d -dimensional polyhedral cone determined by a set of facets $\{F_1, \dots, F_n\}$. We assume that C is pointed at the origin. By $f_i \in (\mathbb{R}^d)^*$ we denote a linear functional defining F_i , i.e. $F_i \subseteq \{\mathbf{x} \in \mathbb{R}^d : f_i(\mathbf{x}) = 0\}$. Let E be an initial extreme ray of C which we find e.g. by solving a generic linear program on C . We compute the extreme rays adjacent to E : first we project C along E by a linear map π . Then, we find the extreme rays of this projected cone $\pi(C)$ (using `cdd`, or applying this procedure recursively). Every extreme ray E_π of $\pi(C)$ corresponds to a two-dimensional face F of C in which E lies. Therefore, there is exactly one more extreme ray E' of C in F . Every $\mathbf{e}' \in E'$ can be written as $\mathbf{e}' = \alpha\mathbf{e} + \beta\mathbf{e}_\pi$, with $\mathbf{e} \in E$, $\mathbf{e}_\pi \in E_\pi$ and some α, β , which we have to compute. This can be done by solving a “two-dimensional linear program”: $f_i(\alpha\mathbf{e} + \beta\mathbf{e}_\pi) = 0$ for all facets of C incident to F , $f_j(\alpha\mathbf{e} + \beta\mathbf{e}_\pi) \geq 0$ for all other facets of C . The key computational step in the procedure above is the computation of the extreme rays of $\pi(C)$. The complexity of this computation is related to the incidence number of E , i.e. the number of facets containing E .

The adjacency decomposition method applies to polyhedral cones having a non-trivial symmetry group:

- (1) Take an initial list of orbits of extreme ray of C .
- (2) Take a representative E of an orbit and finds the extreme rays $(E_i)_{1 \leq i \leq m}$ adjacent to it.
- (3) If some E_i represents a new orbit, then we add it to the list of orbits.
- (4) Finish when all orbits have been treated.

This procedure has two main computational bottlenecks: it can be difficult to identify new orbits, and the incidence of extreme rays can be too high.

Since the symmetry groups of the cones we considered was not too big, the first bottleneck was not a problem: for every new extreme ray we generated the whole orbit.

For dealing with the second bottleneck we used Balinski’s theorem:

Theorem 1. ((1), see e.g. (12))

Let C be a d -dimensional pointed polyhedral cone. Let G be the undirected graph whose vertices are the extreme rays of C and whose edges are the 2-dimensional faces of C . Two vertices E_1, E_2 are connected by an edge F if $E_1, E_2 \in F$. Then, the graph G is $(d - 1)$ -connected, i.e. removal of any $d - 2$ vertices leaves it connected.

Due to Balinski’s theorem, we can replace the criterion “Finish when all orbits have been treated” by “Finish when the number of extreme rays in untreated orbits is lower than $d - 2$ ”.

We applied the adjacency decomposition technique, starting with orbits of lowest incidence. After some time, we found the complete list of orbits. But

still we had to treat the orbits with highest incidence. These very degenerate orbits have usually a particular significance, in our case they correspond to quadratic forms lying in the boundary of $\mathcal{S}_{\geq 0}^d$. In the L -type domain considered, we find out a peculiarity: the number of elements of the orbits with highest incidence is very low, actually lower than $d-2$. Therefore, those orbits cannot disconnect the skeleton graph and so we can stop earlier avoiding the computation of adjacencies of those orbits.

5. DIMENSIONS 1, ..., 5.

In dimension 1 all positive definite quadratic forms are rigid; their Dirichlet-Voronoi polytopes are line segments. In dimension 2 and 3 there are no rigid positive definite quadratic forms because in these dimensions every Dirichlet-Voronoi polytope is a zonotope whence it is a Minkowski sum of line segments. In all these dimensions there is only one non-equivalent primitive L -type domain. In dimension 4 there are three non-equivalent primitive L -type domains, and there is exactly one rigid positive definite quadratic form which is associated to the root lattice D_4 . Its Dirichlet-Voronoi polytope is the 24-cell. In dimension 5 there are 7 rigid positive definite quadratic forms. First, they were enumerated by Baranovskii and Grishukhin (2) by using Engel's list of zone-contracted lattices (4). Our computations confirmed their result.

Now we will argue that these computations do not require much computational effort. All L -type domains up to dimension 4 are simplicial polyhedral cones. As noticed first by Barnes and Trennery (3) (see also the discussion in (8) §13) this does no longer hold in dimension 5 and above. The following table shows how the numbers of facets of primitive L -type domains are distributed among the 222 non-equivalent primitive L -types domains in dimension 5. With n we denote the number of facets and with $L_1(n)$ we denote the number of non-equivalent primitive L -type domains in dimension 5 having exactly n facets.

n	15	16	17	18	19	20	21	22	23	24	25	26	27
$L_1(n)$	62	61	46	17	10	15	6	0	1	3	0	0	1

The next table shows the distribution of the numbers of extreme rays among the 222 non-equivalent primitive L -type domains in dimension 5. With n we denote the number of extreme rays and with $L_2(n)$ we denote the number of non-equivalent primitive L -type domains having exactly n extreme rays.

n	15	16	17	18	19	20	21	22	23	24	25	26
$L_2(n)$	62	84	13	5	33	13	6	0	0	0	0	6

The following table shows how the ranks of the extreme ray are distributed among the 222 non-equivalent L -type domains in dimension 5. By n we denote the number of extreme rays of and by $R_k(n)$ we denote the number of non-equivalent L -type domains having exactly n extreme rays containing positive semidefinite quadratic forms of rank k , $k \in \{1, 4, 5\}$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$R_1(n)$	0	0	0	0	0	0	0	0	0	0	135	58	24	3	1	1	0
$R_4(n)$	55	49	92	19	0	7	0	0	0	0	0	0	0	0	0	0	0
$R_5(n)$	2	12	29	38	56	14	13	17	17	8	4	0	6	0	0	0	6

By our computations we found out that for $d \leq 5$ the graph of primitive L -type domains (vertices = primitive L -type domains, edges = facets between primitive L -type domains) is an infinite tree on which the group $\mathrm{GL}_d(\mathbb{Z})$ acts. The graph of primitive L -type domains is a tree if and only if it has no cycle and therefore if and only if every ridge contains at least one degenerate form. Is this always the case? We think that the answer is “No” even for $d = 6$. We also think that there is a primitive L -type domain whose extreme rays are all non-degenerate.

6. DIMENSION 6

We consider here two different primitive L -type of 6-dimensional positive semidefinite quadratic forms, which were considered by the second author in (10).

6.1. The Cone C_1 . The cone C_1 is a primitive L -type domain of 6-dimensional positive semidefinite quadratic forms. It has 130 facets and we conjecture that there is no L -type domain of 6-dimensional positive semidefinite quadratic forms having more facets. The automorphism group of C_1 has order 1920. Using the adjacency decomposition method we computed that C_1 has 7,145,429 extreme rays in 4,440 orbits. It is worthwhile to note that among the 4,440 orbits there are two orbits which are equivalent under the group $\mathrm{GL}_6(\mathbb{Z})$, and that there is one orbit which does contain forms of rank 5, so that we found altogether 4,438 non-equivalent (under the group $\mathrm{GL}_6(\mathbb{Z})$) rigid positive definite quadratic forms.

6.2. The Cone C_2 . The cone C_2 contains the 6-dimensional positive definite quadratic form associated to the best known 6-dimensional lattice covering (10). It has 100 facets and its automorphism group has order 120. Using the adjacency decomposition method we computed that C_2 has 2,257,616 extreme rays in 20,871 orbits which correspond to 20,861 non-equivalent (under the group $\mathrm{GL}_6(\mathbb{Z})$) rigid positive definite quadratic forms. Three orbits correspond to positive semidefinite quadratic forms being not positive definite.

6.3. Connection between C_1 and C_2 . In this section we show how C_1 and C_2 are related. Both cones C_1 and C_2 contain the rigid form

$$Q_{E_6^*} = \begin{pmatrix} 4 & 1 & 2 & 2 & -1 & 1 \\ 1 & 4 & 2 & 2 & 2 & 1 \\ 2 & 2 & 4 & 1 & 1 & 2 \\ 2 & 2 & 1 & 4 & 1 & 2 \\ -1 & 2 & 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 2 & 2 & 4 \end{pmatrix}$$

associated to the lattice E_6^* . The automorphism groups of G_i of C_i , $i = 1, 2$ are subgroups of the automorphism group $G = \{T \in GL_6(\mathbb{Z}) : T^t Q_{E_6^*} T = Q_{E_6^*}\}$ of $Q_{E_6^*}$.

The subspace I_1 of all quadratic forms invariant under the group G_1 is spanned by $Q_{E_6^*}$ and R_1 (see below) which is an extreme ray of C_1 . If we intersect the cone C_2 with I_1 we get a two-dimensional cone with extreme rays $Q_{E_6^*}$ and R_2 . The rigid forms R_1 and R_2 are

$$R_1 = \begin{pmatrix} 12 & 3 & 6 & 6 & -3 & 3 \\ 3 & 7 & 4 & 4 & 3 & 2 \\ 6 & 4 & 8 & 3 & 1 & 4 \\ 6 & 4 & 3 & 8 & 1 & 4 \\ -3 & 3 & 1 & 1 & 7 & 3 \\ 3 & 2 & 4 & 4 & 3 & 7 \end{pmatrix} \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 2 & 2 \\ 0 & 2 & 0 & 4 & 2 & 2 \\ 0 & 3 & 2 & 2 & 5 & 3 \\ 0 & 1 & 2 & 2 & 3 & 5 \end{pmatrix}$$

6.4. Further Remarks. It is remarkable that there are only very few instances of arithmetically equivalent extreme rays, which are not equivalent under the symmetry group of the cone. The number of extreme rays of both cones is extremely large. Nevertheless it is interesting to note that the number of non-equivalent extreme rays corresponding to forms which are not positive definite is low. In total, we obtained 25,263 new rigid positive definite quadratic forms in dimension 6.

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